

# ARBITRAGE HEDGING STRATEGY AND ONE MORE EXPLANATION OF THE VOLATILITY SMILE

MIKHAIL MARTYNOV <sup>1</sup>, OLGA ROZANOVA <sup>2</sup>

**ABSTRACT.** We present an explicit hedging strategy, which enables to prove arbitrageness of market incorporating at least two assets depending on the same random factor. The implied Black-Scholes volatility, computed taking into account the form of the graph of the option price, related to our strategy, demonstrates the "skewness" inherent to the observational data.

## 1. INTRODUCTION

This work is an extension of the paper [1], where the arbitrage strategy was constructed for a market with pure correlated assets.

Assume that on a market there exist at least two assets with prices  $S_1(t)$  and  $S_2(t)$  which are random processes dependent on the same Brownian motion. It is known that the market with the asset cannot be arbitrage-free. In [2] the principle is formulated that the market is arbitrage-free if and only if the number of traded assets (excluding the riskless one) does not exceed the number of sources of randomness. In the same book it is shown by means of the martingale approach that the market including several risky assets with prices given by the Itô processes is arbitrage-free if and only if the number of independent Wiener processes is equal or greater then the number of the risky assets.

This theoretical result correlates with well known condition for arbitrage: two assets with identical cash flows do not trade at the same price (e.g.[3]).

In the present paper we prove arbitrageness of the market with assets dependent on the same random factors in an alternative way and construct an explicit hedging strategy using a mathematical tool, describing a formation of contrast structures of step type in solutions of semi-linear parabolic equations.

For the sake of definiteness we consider the call option.

## 2. INITIAL-BOUNDARY PROBLEM FOR AN ANALOG OF THE BLACK- SCHOLES EQUATION

Let us consider a financial instrument with the price  $V = V(S_1, S_2, t)$  dependent on prices  $S_1, S_2$  of two different assets and assume that the market is arbitrage-free.

Assume that the prices of the assets are given as follows:

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dW, \quad dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dW.$$

---

*Key words and phrases.* step-like contrast structure, semi-linear parabolic equation, arbitrage, option, hedging strategy, volatility smile.

Supported by the special program of the Ministry of Education of the Russian Federation "The development of scientific potential of the Higher School", project 2.1.1/1399.

We mean that the assets are different if their prices satisfy geometrical Brownian motions having at least different parameters of volatility ( $\sigma_1$  and  $\sigma_2$  in our case).

Consider a portfolio consisting from this financial instrument,  $(-\delta_1)$  units of the first asset and  $(-\delta_2)$  units of the second asset. The price of the portfolio  $\Pi(t)$  at a moment  $t$  is

$$\Pi(t) = V - \delta_1 S_1 - \delta_2 S_2.$$

In this case, using the Itô formula, one can find the law for the price of the financial instrument  $V(S_1, S_2, t)$ :

$$dV = \left( V'_t + \mu_1 S_1 V'_{S_1} + \mu_2 S_2 V'_{S_2} + \frac{1}{2} \sigma_1^2 S_1^2 V''_{S_1 S_1} + \frac{1}{2} \sigma_2^2 S_2^2 V''_{S_2 S_2} + \sigma_1 \sigma_2 S_1 S_2 V''_{S_1 S_2} \right) dt + \left( \sigma_1 S_1 V'_{S_1} + \sigma_2 S_2 V'_{S_2} \right) dW.$$

A change of the portfolio price is written as  $d\Pi = dV - \delta_1 dS_1 - \delta_2 dS_2$ , the arbitrage-free condition leads to the condition  $d\Pi = r(t)\Pi dt = r(t)(V - \delta_1 S_1 - \delta_2 S_2) dt$ , where  $r(t)$  - is the spot interest rate. In what follows we set  $r(t) \equiv r = \text{const}$ . We equate the right hand sides of both expressions for  $d\Pi$  and substitute the expression for  $dV$ . Thus, we get

$$\begin{aligned} (V'_t + \mu_1 S_1 V'_{S_1} + \mu_2 S_2 V'_{S_2} + \frac{1}{2} \sigma_1^2 S_1^2 V''_{S_1 S_1} + \frac{1}{2} \sigma_2^2 S_2^2 V''_{S_2 S_2} + \sigma_1 \sigma_2 S_1 S_2 V''_{S_1 S_2} - \delta_1 \mu_1 S_1 - \delta_2 \mu_2 S_2) dt \\ + \left( \sigma_1 S_1 V'_{S_1} + \sigma_2 S_2 V'_{S_2} - \delta_1 \sigma_1 S_1 - \delta_2 \sigma_2 S_2 \right) dW = r(V - \delta_1 S_1 - \delta_2 S_2) dt. \end{aligned}$$

Equate coefficients at  $dt$  and  $dW$  with zero, we get two equations:

$$\begin{aligned} (V'_t + \mu_1 S_1 V'_{S_1} + \mu_2 S_2 V'_{S_2} + \frac{1}{2} \sigma_1^2 S_1^2 V''_{S_1 S_1} + \frac{1}{2} \sigma_2^2 S_2^2 V''_{S_2 S_2} + \sigma_1 \sigma_2 S_1 S_2 V''_{S_1 S_2} - \\ - \delta_1 \mu_1 S_1 - \delta_2 \mu_2 S_2) = r(V - \delta_1 S_1 - \delta_2 S_2), \\ \sigma_1 S_1 V'_{S_1} + \sigma_2 S_2 V'_{S_2} - \delta_1 \sigma_1 S_1 - \delta_2 \sigma_2 S_2 = 0. \end{aligned}$$

The second equation gives

$$\delta_1 = V'_{S_1} + \frac{\sigma_2 S_2}{\sigma_1 S_1} V'_{S_2} - \delta_2 \frac{\sigma_2 S_2}{\sigma_1 S_1} \quad (1)$$

Substitute this expression for  $\delta_1$  in the first equation. We get

$$\begin{aligned} V'_t + \mu_1 S_1 V'_{S_1} + \mu_2 S_2 V'_{S_2} + \frac{1}{2} \sigma_1^2 S_1^2 V''_{S_1 S_1} + \frac{1}{2} \sigma_2^2 S_2^2 V''_{S_2 S_2} + \sigma_1 \sigma_2 S_1 S_2 V''_{S_1 S_2} - \delta_2 \mu_2 S_2 - \\ - \mu_1 S_1 \left( V'_{S_1} + \frac{\sigma_2 S_2}{\sigma_1 S_1} V'_{S_2} - \delta_2 \frac{\sigma_2 S_2}{\sigma_1 S_1} \right) = r \left( V - S_1 \left( V'_{S_1} + \frac{\sigma_2 S_2}{\sigma_1 S_1} V'_{S_2} - \delta_2 \frac{\sigma_2 S_2}{\sigma_1 S_1} \right) - \delta_2 S_2 \right). \end{aligned}$$

It can be readily shown that

$$\begin{aligned} V'_t + \frac{1}{2} \sigma_1^2 S_1^2 V''_{S_1 S_1} + \frac{1}{2} \sigma_2^2 S_2^2 V''_{S_2 S_2} + \sigma_1 \sigma_2 S_1 S_2 V''_{S_1 S_2} + r S_1 V'_{S_1} \\ + S_2 \left( \mu_2 - \mu_1 \frac{\sigma_2}{\sigma_1} + r \frac{\sigma_2}{\sigma_1} \right) V'_{S_2} + \delta_2 S_2 \left( \mu_1 \frac{\sigma_2}{\sigma_1} - \mu_2 - r \frac{\sigma_2}{\sigma_1} + r \right) - rV = 0. \quad (2) \end{aligned}$$

Let us note that for the construction of the arbitrage hedging strategy it is important that the coefficient at  $\delta_2$  in equation (2) does not vanish. Thus, proving the arbitrageness of the market we show by contradiction that for an arbitrage-free market  $\mu_1 \frac{\sigma_2}{\sigma_1} - \mu_2 - r \frac{\sigma_2}{\sigma_1} + r = 0$ . This means that at a arbitrage-free market the costs of risk  $\frac{\mu_i - r}{\sigma_i}$  for assets  $S_1$  and  $S_2$  coincide. This unobvious fact can be proved differently by means of martingale approach [2].

We note that if one sets  $\delta_2 = 0$  (excluding a dependence of  $\Pi$  on  $S_2$ ), then (2) takes the form of the standard Black-Scholes equation ([4]).

Assume that a buyer of the option does not know that the seller is going to get an additional asset to take part in hedging and therefore he is oriented to the option price, found by the standard Black-Scholes formula. Therefore we set an initial-boundary problem for equation (2), imitating the Cauchy problem for the standard Black-Scholes equation. We denote the solution of the latter one as  $\bar{V}(t, S_1)$  and set the "final" equation  $\bar{V}(S_1, T) = (S_1 - X)^+$ , where  $(S_1 - X)^+ = \max(S_1 - X, 0)$ ,  $X = \text{const} > 0$ . As follows from explicit formula for the solution of this problem, at any moment of time  $t \in [0, T]$  we have  $\bar{V}(0, t) = 0$ , and  $\bar{V}(S_1, t) = S_1 - Xe^{-r(T-t)}(1 - o(\frac{1}{S_1}))$ , at  $S_1 \rightarrow +\infty$ . Let us choose a large positive number  $K_+$  and large by modulus negative number  $K_-$ . We denote

$$S_{\pm} = e^{\frac{K_{\pm}}{\alpha} - c(T-t)}, \quad (3)$$

where  $\alpha > 0$  and  $c$  are constants, which will be chosen later,  $t \in [0, T]$ . It is clear that  $S_- \rightarrow 0$  and  $S_+ \rightarrow +\infty$  as  $|K_{\pm}| \rightarrow \infty$ . We choose functions  $g_{\pm}(S_{\pm}, t) = \bar{V}(S_{\pm}, t)$ . Thus,  $g_-(S_-, t) = o(S_-)$ , as  $S_- \rightarrow 0$  and  $\bar{V}(S_+, t) = S_+ - Xe^{-r(T-t)}(1 - o(\frac{1}{S_+}))$ , as  $S_+ \rightarrow +\infty$ .

So, for any how many large by modulus positive number  $K_+$ , negative number  $K_-$  and any  $S_2 > 0$  we get the initial-boundary problem for equation (2):

$$V(S_1, S_2, T) = (S_1 - X)^+, \quad (4)$$

$$V(S_-, S_2, t) = g_-(S_-, t), \quad V(S_+, S_2, t) = g_+(S_+, t). \quad (5)$$

We can justify the change from the semi-axis  $S_1 > 0$  to the segment  $[S_-, S_+]$  by including in the terms of contract a condition on cancelation of the contract if the price oversteps the limits specified beforehand within a time  $t \in [0, T]$  (the price corridor can be arbitrary large).

### 3. REDUCING (2) TO SEMI-LINEAR PARABOLIC EQUATION

Let us perform several changes of independent and dependent variables of problem (2)-(5) and reduce it to an initially-boundary problem for the heat equation. We do not write the respective initial-boundary problem at every step, only list the changes performed. We note that these change analogous in outline to the changes that reduce the Black-Scholes equation to the heat equation. Nevertheless, there are some distinctions.

We make the change of the time direction  $\tau = T - t$ ; the change of independent variables  $x_1 = \alpha_1 \ln S_1$ ,  $x_2 = \alpha_2 \ln \frac{S_2^{\sigma_1}}{S_1^{\sigma_2}}$ , where  $\alpha_1, \alpha_2 > 0$  are arbitrary constants; the change of dependent variable  $V(x_1, x_2, \tau) = e^{-r\tau}U(x_1, x_2, \tau)$ ; the shift  $y_1 = x_1 + c_1\tau$ ,  $y_2 = x_2 + c_2\tau$ , where  $c_1 = \alpha_1(r - \frac{1}{2}\sigma_1^2)$ ,  $c_2 = \alpha_2(\mu_2\sigma_1 - \mu_1\sigma_2 + \frac{1}{2}\sigma_1\sigma_2(\sigma_1 - \sigma_2))$ .

If in (3) we choose  $\alpha = \alpha_1$  and  $c = c_1/\alpha$ , then we obtain the following initial-boundary problem

$$\begin{aligned} \frac{1}{2}\sigma_1^2\alpha_1^2U''_{y_1y_1} + \delta_2e^{r\tau + \frac{y_2 - c_2\tau}{\alpha_2\sigma_1} + \frac{(y_1 - c_1\tau)\sigma_2}{\alpha_1\sigma_1}} \left( \mu_1\frac{\sigma_2}{\sigma_1} - \mu_2 - r\frac{\sigma_2}{\sigma_1} + r \right) &= U'_{\tau}, \\ U(y_1, y_2, 0) &= \left( e^{\frac{y_1|_{\tau=0}}{\alpha_1}} - X \right)^+, \quad y_1 \in [K_-, K_+], \quad y_2 \in \mathbb{R}, \\ U(K_-, y_2, \tau) &= e^{r\tau}g_-(S_-, \tau), \quad U(K_+, y_2, \tau) = e^{r\tau}g_+(S_+, \tau). \end{aligned} \quad (6)$$

We note that we can consider the variable  $y_2$  as a parameter, since in equation (6) there is not derivatives with respect to it, but the dependence of  $y_2$  remains.

We introduce the notations  $\varepsilon^2 = \frac{1}{2}\sigma_1^2\alpha_1^2$ ,  $U_0(y_1, \varepsilon) = \left(e^{\frac{y_1|_{\tau=0}}{\alpha_1}} - X\right)^+$ ,  
 $F(y_1, \tau, \varepsilon) = -e^{r\tau + \frac{y_2 - c_2\tau}{\alpha_2\sigma_1} + \frac{(y_1 - c_1\tau)\sigma_2}{\alpha_1\sigma_1}} \left(\mu_1 \frac{\sigma_2}{\sigma_1} - \mu_2 - r \frac{\sigma_2}{\sigma_1} + r\right)$ . Since the variable  $\delta_2$ , corresponding to a share of the second asset in the riskless portfolio, can be chosen arbitrary, then we set  $\delta_2 = U(U - A)(U - B)(F(y_1, \tau, \varepsilon))^{-1}$ , where  $A$  and  $B$  are certain constants to be defined below. Such choice of  $\delta_2$  leads to a problem

$$\begin{aligned} \varepsilon^2 U''_{y_1 y_1} - U'_\tau &= f(U), \\ U(y_1, 0, \varepsilon) &= U_0(y_1, \varepsilon), \quad y_1 \in \mathbb{R}, \\ U(K_-, y_2, \tau) &= e^{r\tau} g_-(S_-, \tau), \quad U(K_+, y_2, \tau) = e^{r\tau} g_+(S_+, \tau). \end{aligned} \quad (7)$$

where  $f(U) = U(U - A)(U - B)$ .

Let us approximate the boundary conditions taking into account the fact that the value of  $K$  is large and  $\tau$  is bounded. Note that  $e^{r\tau} g_-(S_-, \tau) \rightarrow 0$  as  $|K_-| \rightarrow \infty$  and  $e^{r\tau} g_+(S_+, \tau) = e^{\frac{K_+}{\alpha} + \frac{\sigma^2 \tau}{2}} - X$ . Then under additional assumption  $\sigma^2 T \ll 1$  we exclude the dependence on time in the boundary conditions:

$$U(K_-, y_2, \tau) = 0, \quad U(K_+, y_2, \tau) = e^{\frac{K_+}{\alpha_1}} - X. \quad (8)$$

Since in the expression  $\varepsilon^2 = \frac{1}{2}\sigma_1^2\alpha_1^2$  the parameter  $\alpha_1$  is arbitrary, we can make  $\varepsilon$  as small as we want.

#### 4. CONDITIONS FOR FORMATION OF THE STEP-LIKE CONTRAST STRUCTURE

Let us outline known results on conditions of formation of the step-like contrast structure [6]. Consider the following initial-boundary problem

$$\begin{aligned} \varepsilon^2 u''_{xx} - u'_t &= f(u, x, \varepsilon), \quad (x, t) \in D \times (0, +\infty), \\ u(a, t, \varepsilon) &= g_a, \quad u(b, t, \varepsilon) = g_b, \quad t \in (0, +\infty), \\ u(x, 0, \varepsilon) &= u_0(x, \varepsilon), \quad x \in \bar{D}, \end{aligned} \quad (9)$$

where  $\varepsilon > 0$  is a small parameter,  $D \equiv (a, b)$ ,  $g_a, g_b$  are constants.

Assume that the function  $f$  satisfies the following conditions.

(A1). There exist functions  $\bar{\omega}$  and  $\hat{\omega}$  from  $C^2(\bar{D})$  such that  $\bar{\omega} < \hat{\omega}, x \in \hat{D}$ , and in the domain  $\Omega = \{(u, x) : \bar{\omega} \leq u \leq \hat{\omega}, x \in \hat{D}\}$  the function  $f(u, x, 0)$  vanishes only on the curves  $u = \varphi_i(x), i = 0, 1, 2$ , moreover,

$$\bar{\omega} < \varphi_1(x) < \varphi_0(x) \equiv 0 < \varphi_2(x) < \hat{\omega}, \quad x \in \hat{D},$$

$$f_u(\varphi_i(x), x, 0) > 0, i = 1, 2; \quad f_u(\varphi_0(x), x, 0) < 0, \quad x \in \hat{D};$$

Assume that  $f(u, x, \varepsilon)$  is sufficiently smooth function in the domain  $\Omega_1 \times [0, \varepsilon_0]$ , where  $\Omega_1$  contains  $\Omega$  and  $\varepsilon_0 > 0$  is an arbitrary number.

We set  $\varphi_0 \equiv 0$  for the sake of simplicity only. It is not difficult to reformulate all results for the case  $\varphi_0 \not\equiv 0$ .

We introduce a function  $J(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(u, x, 0) du$  and make the following assumption.

(A2). There exists a point  $x_0 \in D$  such that  $J(x_0) = 0, \frac{dJ}{dx}(x_0) < 0$ .

(A3). The following inequalities take place:  $\varphi_1(a) < g_a < \varphi_2(a)$ ,  $\varphi_1(b) < g_b < \varphi_2(b)$ ,

$$\int_{\varphi_1(a)}^y f(u, a, 0) du > 0, \quad y \in (\varphi_1(a), g_a], \quad \int_{\varphi_2(b)}^y f(u, b, 0) du > 0, \quad y \in (g_b, \varphi_2(b)].$$

Under conditions (A1) – (A3) for sufficiently small  $\varepsilon$  there exists a stationary solution  $u_s(x, \varepsilon)$  to the boundary problem having an internal transition layer in a neighborhood of a point  $x_0$  such that

$$\lim_{\varepsilon \rightarrow 0} u_s(x, \varepsilon) = \begin{cases} \varphi_1(x), & x \in (a, x_0) \\ \varphi_2(x), & x \in (x_0, b). \end{cases} \quad (10)$$

The solution of this kind are called contrast step-like structure (CSLS).

It is known that under conditions (A1) – (A3) the CSLS solution  $u_s(x, \varepsilon)$  is an asymptotically stable solution to the boundary problem. There arises a question on a set of initial values  $u_0(x, \varepsilon)$ , which lead to a formation of the contrast structure  $u_s(x, \varepsilon)$  as  $t \rightarrow +\infty$ . In other words, which is a domain of influence of this solution? Let us give the definition of the global domain of influence according to [6].

Let the boundary problem have a stationary solution  $u_\varepsilon(x) \in C^2(\bar{D})$  at  $\varepsilon \in (0, \varepsilon']$ , where  $\varepsilon' > 0$  is a certain number.

**Definition 1.** We call  $G(u_\varepsilon)$  the global domain of influence of a stationary solution  $u_\varepsilon(x)$  of the boundary problem if  $G(u_\varepsilon)$  contains functions  $u_0(x, \varepsilon)$  having the following property: there exists  $\varepsilon'' \in (0, \varepsilon']$  such that at  $\varepsilon \in (0, \varepsilon'']$  there is a solution  $u_\varepsilon(x, t) \in C^{1,0}(\bar{D} \times [0, +\infty)) \cap C^{2,1}(\bar{D} \times (0, +\infty))$  of the initial-boundary problem and  $\lim_{t \rightarrow +\infty} \|u_\varepsilon(x, t) - u_\varepsilon(x)\|_{C(\bar{D})} = 0$ .

Let  $f$  satisfy additional conditions:

(A4). The set of all points  $x$  such that  $J(x) = 0$  consists of finite number of segments or points.

(A5).  $u_0(x, \varepsilon) \equiv u_0(x) \in C_B^2(\bar{D}) \equiv \{v(x) \in C^2(\bar{D}) : v(a) = g_a, v(b) = g_b\}$

$$\varphi_1(x) \leq u_0(x) \leq \varphi_2(x), x \in \bar{D}.$$

(A6).  $\exists x^{(-)} \in (a, x_0)$  and  $\exists x^{(+)} \in (x_0, b)$  such that

$$u_0(x^{(-)}) < \varphi_0(x) \quad u_0(x) < \varphi_0(x) \quad x \in [a, x_0), \quad J(x) \leq 0,$$

$$u_0(x^{(+)}) > \varphi_0(x) \quad u_0(x) > \varphi_0(x) \quad \text{at all points } x \in (x_0, b], \quad \text{where } J(x) \geq 0.$$

The main result of [6] is the following theorem.

**Theorem 1.** Let conditions (A1) – (A3) hold. Then for sufficiently small  $\varepsilon$  there exists a stationary solution  $u_s(x, \varepsilon)$  of the boundary problem (9) from  $C^2(\bar{D})$ , having form of the contrast step-like structure, satisfying limit equation (10).

Moreover, let condition (A4) hold. Then the following is true:

1. If a function  $u_0(x, \varepsilon) \equiv u_0(x)$  satisfies (A5), (A6), then it falls in  $G(u_s)$ .
2. If  $\exists \varepsilon_1 > 0$  such that at  $\varepsilon \in (0, \varepsilon_1]$  a function  $u_0(x, \varepsilon)$  satisfies condition (A5) and there exist functions  $\bar{u}_0(x)$  and  $\hat{u}_0(x)$ , satisfying (A5), (A6) and such that

$$\bar{u}_0(x) < u_0(x, \varepsilon) < \hat{u}_0(x), \quad x \in \bar{D}, \quad \varepsilon \in (0, \varepsilon_1],$$

then  $u_0(x, \varepsilon)$  belongs to  $G(u_s)$ .

## 5. FORMATION OF SLCS IN PROBLEM (7)

We re-formulate problem (7) with boundary conditions, changed to (8) (we write  $x$  instead of  $y_1$  and  $u$  instead of  $U$ ):

$$\begin{aligned} \varepsilon^2 u''_{xx} - u'_t &= f(u), \\ u(x, 0) &= \left( e^{\frac{x}{\alpha_1}} - X \right)^+, \\ u(a, t) &= 0, \\ u(b, t) &= e^{\frac{K_+}{\alpha_1}} - X, \end{aligned} \tag{11}$$

where  $f(u) = u(u - A)(u - B)$ ,  $0 < A < B$ ,  $t \in [0, T]$ ,  $x \in [a, b]$ ,  $a = K_-$ ,  $b = K_+$ .

Let us apply the results of the CSLS theory outlined in Sec. 4 to problem (11). Condition **(A1)** holds evidently. From condition **(A2)** we get a relation between constants  $A$  and  $B$ . Since  $\int_0^B f(u) du = \int_0^B u(u - A)(u - B) du = 0$ , then  $B = 2A$ .

Thus, the condition **(A4)** holds since  $J(x) \equiv 0$  for all segment  $[a, b]$ . It remains to find  $A$  to satisfy **(A3)**. We take  $A = \frac{u(K_+, t)}{2} = \frac{e^{\frac{K_+}{\alpha_1}} - X}{2}$ . The point of transfer can be found as

$$x^0 = a + (b - a) \frac{\sqrt{f_u(\varphi_2)}}{\sqrt{f_u(\varphi_2)} + \sqrt{f_u(\varphi_1)}} = K_- + (K_+ - K_-) \frac{\sqrt{f_u(2A)}}{\sqrt{f_u(2A)} + \sqrt{f_u(0)}} = \frac{K_- + K_+}{2}$$

(see Butuzov, Vassilieva).

Thus, the contrast structure in the stationary problem has a form

$$\lim_{\varepsilon \rightarrow 0} u(x, \varepsilon) = \begin{cases} 0, & K_- < x < 0, \\ e^{\frac{K_+}{\alpha_1}} - X, & 0 < x < K_+. \end{cases}$$

Condition **(A5)** holds due to the relation  $u(x, 0) \equiv (\varphi_2(x), 0)^+$ , and condition **(A6)** holds due to  $u(x, 0) \equiv \frac{1}{2}\varphi_0(x)$ . Since all conditions of Theorem 1 take place, then in the problem (11) a contrast step-like structure arises.

## 6. ARBITRAGE HEDGING STRATEGY

We note that we can choose the values  $K_-$ ,  $K_+$  arbitrary. Therefore, increasing  $S_+$  one can always shift the transition point  $S_0 = \sqrt{S_+ S_-}$  a little to the right of the strike price  $X$ . This means that we can choose such hedging strategy  $(\delta_1, \delta_2)$ , that the option price can be negligibly small initially.

The option price, found by the classical Black-Scholes formula, it is greater initially than at the moment  $T$ . But if we apply the hedging strategy involving the second asset, then we get that the initial option price (at  $S < S_0$ ) is negligible comparing with its price at the moment of exercise. This gives evidently a possibility of arbitrage upon conclusion of contract. Thus, we get a contradiction with the assumption with a non-arbitrageness of the market.

The hedging strategy  $(\delta_1, \delta_2)$  has a form:

$$\delta_1 = V'_{S_1} + \frac{\sigma_2 S_2}{\sigma_1 S_1} V'_{S_2} - \delta_2 \frac{\sigma_2 S_2}{\sigma_1 S_1}, \quad \delta_2 = V(V - (S_+ - X)/2)(V - (S_+ - X)) \frac{1}{S_2(\mu_2 - \mu_1 \frac{\sigma_2}{\sigma_1})}.$$

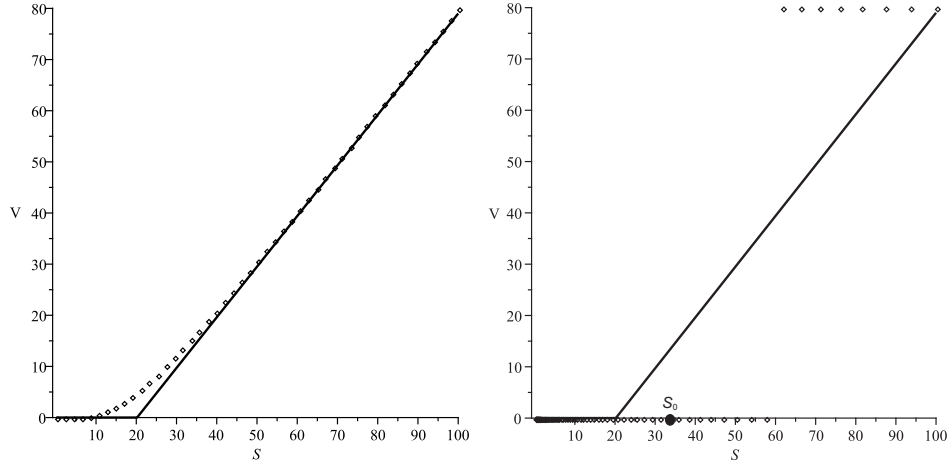
## 7. NUMERICAL SOLUTION OF PROBLEM (11)

Let us construct the contrast structure in our problem numerically. In particular, we understand how quick the structure forms. We use the Crank - Nicolson method and the marching. Recall that we seek for the solution to the problem (11) in the domain  $[K_-, K_+] \times [0, T]$ .

We choose the following parameters:  $S_- = 0.1$ ,  $S_+ = 100$ ,  $N = 100$ ,  $\tau = 2 \cdot 10^{-4}$ ,  $\alpha_1 = 1$ ,  $\sigma_1 = 0.02$ ,  $X = 20$ . Then the transfer point is  $S_0 = 31.6$ , the size of the step is  $A = 40$ .

Pic.1 presents the "final" function  $V(S_1, S_2, T)$  (solid line), and the numerical solution  $V(S_1, S_2, t)$  at  $t = 0$  (dashed line). Even at  $T = 0.25$  (the exercise time equal to 3 months) the graph of solution is step-like.

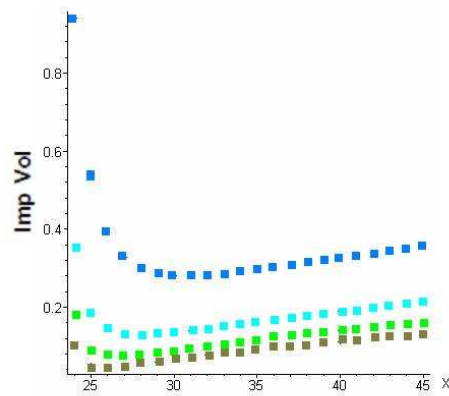
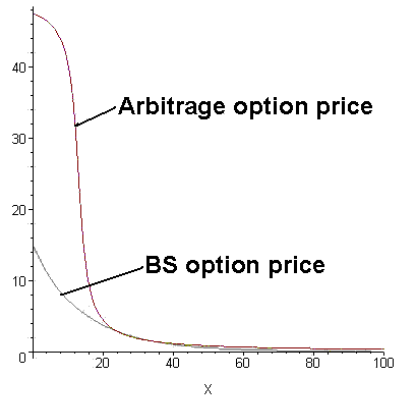
Pic.2 presents the graph of the same function  $V = V(S_1, t)$   $t = 0$  (points) and at  $t = T$  (solid line) by hedging without using the second asset (.g.  $\delta_2 = 0$ ) as in classical Black-Scholes model.



## 8. "VOLATILITY SMILE" AS A MANIFESTATION OF ARBITRAGENESS OF MARKET

Below we present the graphs of the option price in dependence on the strike price  $X$  (fixed spot price and the time of exercise) for the arbitrage-free Black-Scholes hedging strategy and arbitrage hedging strategy considered in this paper (Pic.3). Pic.4 presents the graphs for the dependence of volatilities on the strike price computes basing on the Black-Scholes formula for different times of exercise. The graphs clearly demonstrate the "skew smile", moreover, the skew increases as the exercise time gets smaller, as is in compliance with observational data (e.g. [7]). It is interesting that for explanation of this phenomenon we do not need to engage the idea of stochastic volatility as they usually do. We note that the

phenomena of "smile" of volatility near the strike price is observed since 1987, where the amendment to the Glass-Steagall Act allowed to invest the bank capitals in derivatives. Since the asset and its derivative are correlated, it inevitably gives an arbitrage possibility and therefore the main conditions for the Black-Scholes formula can not be satisfied.



## REFERENCES

- [1] Martynov, M.A. Construction of an arbitrage hedging strategy in a market with assets depending on the same random factor. Moscow University Mathematics Bulletin. 2010. **65**. 238-243.
- [2] Bjork, T. Arbitrage Theory in Continuous Time // Oxford University Press. 2003.
- [3] <http://en.wikipedia.org/wiki/Arbitrage>
- [4] Black F., Scholes M. The Pricing of Options and Corporate Liabilities // Journal of Political Economy. 1973. **81**. 659-683.



- [5] Vasil'eva, A.B., Butuzov, V.F., Kalachev, L.V. The boundary function method for singular perturbation problems. SIAM Studies in Applied Mathematics, 14, 1995.
- [6] Butuzov, V. F.; Kryazhinsky, S. A.; Nedel'ko, I. V. On the global domain of influence of stable steplike contrast structures in the Dirichlet problem. Comput. Math. Math. Phys. 44 (2004), no. 6, 9851006.
- [7] Derman, E. The Problem of the Volatility Smile. Talk at the Euronext Options Conference, Amsterdam, May 26, 2003, [http://www.ederman.com/new/docs/euronextvolatility\\_smile.pdf](http://www.ederman.com/new/docs/euronextvolatility_smile.pdf)

<sup>(1, 2)</sup> MATHEMATICS AND MECHANICS FACULTY, MOSCOW STATE UNIVERSITY, MOSCOW 119992, RUSSIA

*E-mail address*, <sup>1</sup>: [mikhailmartynov@gmail.com](mailto:mikhailmartynov@gmail.com)

*E-mail address*, <sup>2</sup>: [rozanova@mech.math.msu.su](mailto:rozanova@mech.math.msu.su)